# Harmonic Maps and the Optimal
# Design of Mechanisms

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### Abstract

In this paper we address certain problems in the design and analysis of kinematic chains using the theory of harmonic maps. We show that this theory provides interesting results on the treatment of optimal geometries for both the redundant and non-redundant cases and provides a reliable numerical criterion for dexterity.

## 1   Introduction

Until recently much of the kinematic design and performance evaluation of robot manipulators has been based largely on experience and intuition, with varying notions on what constitutes an optimal design. Although it is clear that no single design exists which is optimal for all tasks, it is still possible to associate with mechanisms a certain sense of *coordination* or *dexterity*, physical qualities which are normally ascribed to humans. What is often lacking is a formal, mathematically precise way of characterizing these qualitative observations. The benefits of such a tool for kinematic analysis are, of course, apparent and numerous. It is generally agreed, for example, that extra degrees of freedom will enhance the performance of a manipulator, yet methods for exploiting this redundancy remain largely an art. This paper provides an analytic tool for treating such problems in a precise manner.

One of the first kinematic performance criteria proposed the concept of maximal dextrous workspace volume (Roth 1976). This criterion measures the volume of the set of points in the workspace which can be reached with arbitrary orientation of the end-effector. Several variations and applications of the workspace volume criterion have been proposed and investigated by others (Gupta 1986, Vijaykumar 1986). Such criteria are appropriate for manipulators whose task descriptions involve large movements relative to the size of the mechanism. However, just as for human coordination, we can argue that a mechanism's ability to perform complex, fine motions in a smooth and accurate manner is in fact more critical than its workspace volume. This is especially true for general assembly robots which are being asked to perform increasingly complex tasks. Several criteria measuring this dexterity in a local sense have been proposed. Yoshikawa (1985) defines the *manipulability measure* as the scalar quantity $\sqrt{\det J(\theta)J^T(\theta)}$, where $J(\theta)$ is the Jacobian of the forward kinematic map. Klein (1987) investigates this and other related dexterity measures with respect to optimal postures, working points and link lengths for redundant planar mechanisms. These measures all share the common theme of penalizing configurations which are close to singularities. However, the distance from a singular configuration does not necessarily correspond to our intuitive notion of dexterity, and furthermore it is not possible to extend these

measures globally over the entire domain, precisely because these measures become degenerate at the singularities. What is desired, it seems, is a means of measuring dexterity "uniformly" over the entire domain including the singularities.

We show here that the theory of harmonic maps, which is generally traced to the fundamental paper of Eells and Sampson (Eells 1964), provides an interesting analytical tool for investigating the kinematic dexterity of mechanisms. Harmonic maps are the critical points of an integral distortion measure which we claim is a natural measure of our intuitive notion of dexterity. Considering mechanisms to be smooth maps between the Riemannian manifolds of the joint space $M$ and the end-effector space $N$, the functional measures the inherent distortion involved in the mapping of spaces of different curvatures onto one another. The functional used measures the distortion of a map in some uniform average sense by integrating over the entire domain $M$.

Several different interpretations of this distortion measure are possible. A rough physical interpretation is that if $M$ were made of rubber and $N$ of marble, we then wish to map this elastic domain $M$ onto a rigid range $N$. We can therefore associate with each point of $M$ an elastic *tension*; harmonic maps are ones which result in an elastic equilibrium of minimum average tension (Eells 1978). An alternative interpretation involves probability. As is well known, if the Jacobian defines the velocity gain in going from joint space to end-effector space, then its inverse transpose defines the force-torque gain; a large velocity gain results in a small force-torque gain. Therefore by controlling the size of the velocity gains we can control the force-torque gain. Specifically, if we let the velocity vector in joint space be randomly distributed according to a zero mean, unity variance gaussian, and if we define the average velocity gain to be the expected value of the length of the velocity vector in end-effector space, then the harmonic map minimizes the velocity gains, averaged over the entire set of joint values. In this case the average is computed relative to the natural volume measures in both the range and domain; this idea may be compared with the notion of dispersion introduced by Eells (1964).

In section 2 we introduce the distortion measure and introduce the theory of harmonic maps; we need to assume that the reader is familiar with the basic concepts of Riemannian geometry. To gain familiarity with the concepts we consider several examples of simple manipulators in section 3, computing the distortion and optimal link lengths for various cases. These examples illuminate the key features of our distortion measure and suggest ways to address other issues arising in the design and analysis of mechanisms.

## 2 Harmonic Maps

We begin with some geometric preliminaries. Let $M$ and $N$ be Riemannian manifolds of dimension $m$ and $n$, respectively, endowed with the metrics $g_{ij}dx^i dx^j$ and $h_{\alpha\beta}df^\alpha df^\beta$, in local coordinate charts $x = (x^1 \ldots x^n)$ and $f = (f^1 \ldots f^m)$ on $M$ and $N$, respectively. Define the metric tensor $(h^{\alpha\beta})$ to be $(h_{\alpha\beta})^{-1}$. If we consider a mechanism to be a smooth map $f : M \to N$, then we can define its distortion density in local coordinates as

$$d(\mathbf{f}) = \frac{1}{2} h^{\alpha\beta}(\mathbf{x})g_{ij}(\mathbf{f}(\mathbf{x}))\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}$$

where we use the standard summation convention of repeated indices being summed over their entire range. If we denote the Jacobian matrix of the map $f$ by $J$, and the metric tensors $(g_{ij})$ and $(h_{\alpha\beta})$ by $G$ and $H$, respectively, then an alternative representation for the distortion density is $\frac{1}{2} tr\{J^T H J G^{-1}\}$. The total distortion of the map $f$ is then defined to be

$$D(\mathbf{f}) = \int_M d(\mathbf{f})dM$$

where $dM$ is the volume element on $M$. If $f$ is of class $C^2$, the distortion $D(f)$ is finite, and $f$ is a critical point of $D(f)$, then the map $f$ is called *harmonic*. As with finding critical points of other functionals, we can derive a set of Euler equations for harmonic maps. Let $\Gamma^i_{jk}$ denote the Christoffel symbols of the second kind on $N$. Then the Euler equations are of the form

$$\frac{1}{\sqrt{|H|}} \frac{\partial}{\partial x^\alpha}(\sqrt{|H|}\, h^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}) + h^{\alpha\beta}\Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} = 0$$

for $i = 1, 2, \ldots, m$, where once again $H$ denotes the metric tensor $(h_{\alpha\beta})$ and $|H|$ its determinant. We thus obtain a nonlinear elliptic system of $m$ partial differential equations, where the principal part on the left is the Laplace-Beltrami operator on $M$, and the second term is quadratic in the gradient of the solution. For our purposes this is unnecessarily abstract. It is easier for our applications to derive these equations on a problem-by-problem basis using variational principles for multiple integrals, rather than computing the Christoffel symbols on $N$.

Suppose we wish to find the critical points of a functional of the form

$$D(f) = \int_M L(x, f(x), \nabla f(x))dx$$

where

$$\nabla f(x) = (\frac{\partial f^1}{\partial x^1} \cdots \frac{\partial f^1}{\partial x^n} \cdots \frac{\partial f^m}{\partial x^1} \cdots \frac{\partial f^m}{\partial x^n})$$

Assuming $M$ has fixed boundaries, the Euler equations then are

$$\frac{\partial L}{\partial f^i} - \sum_{j=1}^{n} \frac{d}{dx^j}\frac{\partial}{\partial(\frac{\partial f^i}{\partial x^j})}L = 0$$

for $i = 1, \ldots, m$. These equations are an alternative version of the Euler equations discussed in the previous paragraph. Some familiar examples of harmonic maps are the solutions of Laplace's equation for the equilibrium temperature distribution in a plate, and the geodesics on a manifold.

It is instructive to consider simple one-dimensional examples to see that harmonic maps are indeed distortion minimizing maps. For the case $M = N = [0,1]$, the distortion criterion is simply $D(f) = \int_0^1 \dot{f}^2\, dt$, where $f : [0,1] \to [0,1]$ is

constrained to be onto the interval $[0,1]$. The solution is the linear function $f = t$, which inarguably is a minimum distortion map. Another possible choice of distortion measure of a map is its "proximity" to an isometry; recall that $f : M \to N$ is an isometry if $J^T H J = G$, where $G$ and $H$ are, once again, the metric tensors on $M$ and $N$, and $J$ is the jacobian of $f$. We can therefore consider $||J^T H J - G||$, for some suitably defined norm, to be a distortion density. The standard metric

$$||J^T H J - G|| = tr\left\{(J^T H J - G)(J^T H J - G)^T\right\}$$

leads to Euler equations which are fourth-order, however, and just the question of existence of solutions becomes an extremely difficult problem. For practical purposes this measure seems too complicated and unsuitable. However it is interesting to note that for our one-dimensional example above, both distortion measures lead to the same solution, and it is plausible that this may occur for other cases as well. To fix ideas we will now consider a simple but interesting example.

### Example 1 – Harmonic Maps from $T^2$ to $S^2$

Suppose we wish to find a harmonic map from the flat torus $T^2$ to the unit two-sphere $S^2$ embedded in $\Re^3$. Denote local coordinates on $T^2$ by $(x,y)$ and polar coordinates $(\theta,\phi)$ on $S^2$, where $\theta$ is the longitudinal coordinate, and $\phi = 0$ corresponds to the z-axis. Endow the two spaces with their usual Riemannian metrics: $ds^2 = dx^2 + dy^2$ on $T^2$, and $ds^2 = d\phi^2 + \sin^2\phi\, d\theta^2$ on $S^2$. The distortion is then given by

$$D(f) = \frac{1}{2} \int \int_{T^2} ||\nabla\phi||^2 + \sin^2\phi ||\nabla\phi||^2 dx\, dy$$

where $\nabla$ denotes the gradient operator and $||\cdot||$ is the standard Euclidean norm. The Euler equations are

$$\nabla^2\theta + 2\cot\phi < \nabla\theta, \nabla\phi > = 0$$

$$\nabla^2\phi - \sin\phi\cos\phi ||\nabla\theta||^2 = 0$$

If we fix the dimensions of the torus to be $2\pi$ by $2\pi$, then for a double covering of $S^2$ by $T^2$ we require the boundary conditions $\theta(0) = 0$, $\theta(2\pi) = 2\pi$, $\phi(0) = 0$, $\phi(2\pi) = 2\pi$. There is in fact a theorem which states that no single covering harmonic maps exist from $T^2$ to $S^2$ (Eells 1988); we must therefore cover the sphere twice with a "degree two" map. Suppose $\theta$ is the linear function $\theta(y) = y$. The Euler equations will then be satisfied if $\phi$ satisfies the pendulum equation

$$\frac{\partial^2\phi}{\partial x^2} - \sin\phi\cos\phi = 0$$

with boundary conditions $\phi(0) = 0$, $\phi(2\pi) = 2\pi$. The distortion reduces to

$$D(f) = \pi \int_0^{2\pi} (\frac{\partial\phi}{\partial x})^2 + \sin^2\phi\, dx$$

By symmetry, the latter boundary condition to the pendulum equation can be replaced by $\phi(\frac{\pi}{2}) = \frac{\pi}{2}$, and the total distortion is four times the density integrated over this interval.

Note that we were able to reduce the Euler equation to a single pendulum equation. Baird (1984) provides a more general "reduction" theorem describing circumstances under which the Euler equations for a harmonic map between manifolds of constant curvature can be reduced to a single nonlinear equation of the pendulum type.

Similarly, we can look for harmonic maps from $S^2$ to $T^2$. The distortion in this case becomes

$$D(\mathbf{f}) = \int\!\!\int_{S^2} \frac{1}{\sin\phi}\{(\frac{\partial x}{\partial\theta})^2 + (\frac{\partial y}{\partial\theta})^2\}$$

$$+ (\frac{\partial x}{\partial\phi})^2 + (\frac{\partial y}{\partial\phi})^2 \ d\theta \ d\phi$$

and the corresponding Euler equations are

$$\frac{1}{\sin^2\phi}\frac{\partial^2 x}{\partial\theta^2} + \frac{\partial^2 x}{\partial\phi^2} = 0$$

$$\frac{1}{\sin^2\phi}\frac{\partial^2 y}{\partial\theta^2} + \frac{\partial^2 y}{\partial\phi^2} = 0$$

Note that the linear solution $x = \theta$, $y = \phi$ satisfies the equations. This is the standard topography problem, and our solution confirms that the familiar polar coordinates are in fact a distortion minimizing map.

## 3 Applications to Robot Kinematics

**Example 2 – $T^2$ to $S^2$ Revisited**

Any mechanism with two rotary joints has the torus $T^2$ as its joint space. Some of the typical mechanisms of this class have the annulus, the disk and the two-sphere as the range space. We therefore reconsider the $T^2$ to $S^2$ problem from the mechanism perspective. Since there is nothing sacred about the square torus, we leave the dimensions of the torus to be arbitrary for now, requiring only that its area be $4\pi$. Denoting the width of the torus edges by $x_{max}$ and $y_{max}$, and once again assuming the linear solution $\theta(y) = (\frac{2\pi}{y_{max}})y$ for the longitudinal coordinate, we are left with the pendulum equation

$$\frac{\partial^2\phi}{\partial x^2} - (\frac{4\pi^2}{y_{max}^2})\sin\phi\cos\phi = 0$$

with boundary conditions $\phi(0) = 0$ and $\phi(y_{max}) = 2\pi$. We determine the distortion $D(\mathbf{f})$ numerically as a function of the aspect ratio $r = (\frac{y_{max}}{x_{max}})$. We see from figure 1 that as $r$ approaches 0, the distortion decreases to the minimum value of $8\pi$. This is in accordance with a theorem which states that the infimum of the distortion in the class of maps of degree $k$ from $T^2$ to $S^2$ is $|k|\ \mathrm{Vol}(S^2)$, where $k = 2$ for our double-covering map (Eells 1988).

Now suppose that $\phi$ is the linear function $\phi = (\frac{2\pi}{y_{max}})y$, so that the map now corresponds to a conventional physical mechanism. The distortion in this case can be reduced to $D(f) = \pi^2(2r + \frac{1}{r})$. We see from figure 1 that contrary to the harmonic map case, there is an optimal aspect ratio of $\frac{1}{\sqrt{2}}$
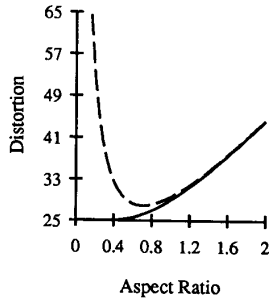


Figure 1: A plot of distortion vs. aspect ratio $y_{max}/x_{max}$ for the map from $T^2$ to $S^2$: the solid line represents the harmonic map, while the dashed line represents the linear map
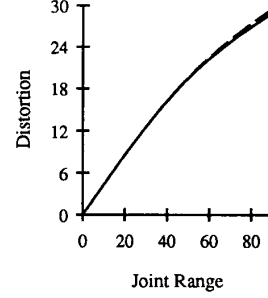


Figure 2: A plot of distortion vs. joint range of $\phi$ for the $T^2$ to $S^2$ problem: the solid line is the harmonic map, while the dashed line represents the linear map

for which the distortion attains a minimum. This optimal aspect ratio, in fact, can be interpreted as the optimal ratio of the maximum velocities for the joint motors of the physical mechanism. Since the mechanism is also a double-covering, the motor controlling the $\theta$ coordinate should have a maximum velocity $\sqrt{2}$ times larger than the maximum velocity of the $\phi$ coordinate motor in order to achieve maximum dexterity. Finally, in figure 2 we plot the difference in the distortion between the harmonic map and the linear mechanism as a function of the joint-range of $\phi$. As we restrict $\phi$ to lie in the range $[\triangle, \pi - \triangle]$, for $\triangle \to \frac{\pi}{2}$ we expect the difference in the distortion for the two cases to go to zero.

**Example 3 – Computing Optimal Link Lengths for Planar Mechanisms**

One of the strengths of our distortion measure is that it is not limited to the case of equal dimensions of the joint and end-effector spaces; in fact, it is defined for any arbitrary dimensions of $M$ and $N$. Redundancy problems can therefore be treated by our measure without encountering any major additional difficulties. As a demonstration of this we compute the optimal link-lengths for an $n$-link planar manipulator based on our distortion criterion. This mechanism can be thought of as a map from the flat $n$-torus $T^n$ to a disk of fixed radius $L$. The kinematic equations are given by

$$x = \sum_{i=1}^{n} L_i \cos (\sum_{j=1}^{i} \alpha_j)$$

$$y = \sum_{i=1}^{n} L_i \sin (\sum_{j=1}^{i} \alpha_j)$$

where the $L_i$ are the link-lengths of the planar manipulator, and $\alpha_i$ are the toroidal angles. We parametrize the disk by polar coordinates $(r, \theta)$ and the flat torus by local coordinates $\alpha = (\alpha_1 \cdots \alpha_n)$. The Riemannian metrics for the flat torus and the disk are then given by $ds^2 = d\alpha_1^2 + d\alpha_2^2 + \ldots + d\alpha_n^2$ and $ds^2 = dr^2 + r^2 d\theta^2$ respectively. The total distortion for the map is then

$$D(r,\theta) = \frac{1}{2}\int_{T^n} ||\nabla r||^2 + r^2 ||\nabla\theta||^2 \ d\alpha$$

Note that since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(\frac{y}{x})$, we have

$$\frac{\partial r}{\partial\alpha_i} = \frac{1}{r}(x\frac{\partial x}{\partial\alpha_i} + y\frac{\partial y}{\partial\alpha_i})$$

$$\frac{\partial \theta}{\partial \alpha_i} = \frac{1}{r^2}(x\frac{\partial y}{\partial \alpha_i} - y\frac{\partial x}{\partial \alpha_i})$$

The distortion is therefore simplified to

$$D(r,\theta) = \frac{1}{2}\int_{T^n}\sum_{i=1}^{n}(\frac{\partial x}{\partial \alpha_i})^2 + (\frac{\partial y}{\partial \alpha_i})^2 d\alpha$$

A straightforward calculation reduces the distortion further to

$$D(r,\theta) = \sum_{k=1}^{n}kL_k^2$$

The optimal link-lengths $(L_1^* \cdots L_n^*)$ which minimize the distortion subject to the constraint that the sum of the link lengths be $L$, can then be derived by solving the linear system

$$\begin{bmatrix} 3 & 1 & \cdots & \cdots & 1 \\ 1 & 4 & \cdots & \cdots & 1 \\ \vdots & & \ddots & & 1 \\ \vdots & & & n & 1 \\ 1 & 1 & \cdots & 1 & n+1 \end{bmatrix}\begin{bmatrix} L_1 \\ \vdots \\ \vdots \\ L_n \end{bmatrix} = \begin{bmatrix} L \\ \vdots \\ \vdots \\ L \end{bmatrix}$$

By rewriting the matrix as

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 3 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & n \end{bmatrix}$$

and applying the well known formula for the inverse of a rank 1 perturbation we immediately get

$$L_k = \frac{L}{k\sum_{j=1}^{n}(\frac{1}{j})}$$

Hence, in terms of proportional lengths we get for the $n$-links the ratio $(1, \frac{1}{2}, \ldots, \frac{1}{n})$, where the proximal link $L_1$ is of length 1. In particular, for the 3-link case we get link lengths 6, 3 and 2 respectively for $L_1$, $L_2$ and $L_3$. This is to be compared with another link length result indicating that for maximum dexterity adjacent limbs are in a ratio of $\sqrt{2}$ to 1, the link lengths decreasing as we move away from the base.

We have thus far assumed the flat torus $T^n$ to have fixed widths $2\pi$. If we restrict our attention to 2-link mechanisms and leave the torus dimensions arbitrary, then we can once again compute the distortion as a function of the aspect ratio $r$ and the link lengths $L_1$ and $L_2$. The distortion $D(f)$ can be reduced to

$$D(f) = L_1^2 + L_2^2(r + \frac{1}{r})$$

for which the optimal aspect ratio

$$r = \frac{L_2}{(L_1^2 + L_2^2)^{\frac{-1}{2}}}$$

Fixing the total link length $L_1 + L_2$ to be 1, the optimal link lengths and aspect ratio are found to be $L_1 = 0.561$, $L_2 = 0.469$, $r = 0.641$.

## Example 4 – Computing Optimal Link Lengths for a Shoulder and Elbow Joint

We now compute the optimal lengths for a 2-link shoulder and elbow joint. The standard shoulder and elbow joint can be

considered as a map $f : S^2 \times S^1 \to B^3$, where $B^3 \in \Re^3$ is the unit ball in 3 dimensions. Choose local coordinates $(x\ y\ z)$ on $B^3$ and $(\theta\ \phi\ \psi)$ on $S^2 \times S^1$, where $(\theta\ \phi)$ represent standard spherical coordinates, and $\psi$ is the angle which link 2 forms with respect to the radial extension of link 1. Let $0 \le \theta$, $\psi \le 2\pi$ and $\triangle \le \phi \le \pi - \triangle$ for some parameter $\triangle$. The map $f$ can then be expressed in coordinates as

$$\begin{aligned} x &= L(\psi)\cos\theta\sin\alpha \\ y &= L(\psi)\sin\theta\sin\alpha \\ z &= L(\psi)\cos\alpha \end{aligned}$$

where

$$\begin{aligned} L^2(\psi) &= L_1^2 + L_2^2 + 2L_1L_2\cos\psi \\ \alpha(\phi,\psi) &= \phi + \tan^{-1}(\frac{L_2\sin\psi}{L_1 + L_2\cos\psi}) \end{aligned}$$

The distortion density associated with the map $f$ is given by

$$\begin{aligned} d(f) =\ & \tfrac{1}{2}\{(\frac{\partial x}{\partial \phi})^2 + (\frac{\partial y}{\partial \phi})^2 + (\frac{\partial z}{\partial \phi})^2 \\ & + (\frac{\partial x}{\partial \psi})^2 + (\frac{\partial y}{\partial \psi})^2 + (\frac{\partial z}{\partial \psi})^2 \\ & + \tfrac{1}{\sin^2\phi}((\frac{\partial x}{\partial \theta})^2 + (\frac{\partial y}{\partial \theta})^2 + (\frac{\partial z}{\partial \theta})^2)\} \end{aligned}$$

Noting that the volume form on $S^2 \times S^1$ is $dV = \sin\phi\ d\theta\ d\phi\ d\psi$, the distortion can be written as

$$D(f) = 8\pi^2\{1 + L_1^2 + \frac{\cos\triangle}{2}\} + 2\pi^2 L_2^2 \int_{\triangle}^{\pi-\triangle}\frac{\cos^2\phi}{\sin\phi}d\phi$$

Note that because of the singularities at $\phi = 0$ and $\phi = \pi$, the last term is not integrable for $\triangle = 0$. In fact, the distortion is infinity at these points. If we limit the joint range of $\phi$ to exclude the singularities, however, the distortion is finite, and it is then possible to compute optimal link lengths. Fixing the total link length $L_1 + L_2$ to be 1, figure 3 plots the optimal length of link 1 as a function of the joint range of $\phi$. Note that link 1 will always be longer than link 2 for useful joint ranges of $\phi$. In particular, for optimal link lengths in the ratio $L_1 : L_2 = \sqrt{2}$, we see that the corresponding joint range for $\phi$ is roughly 165°.
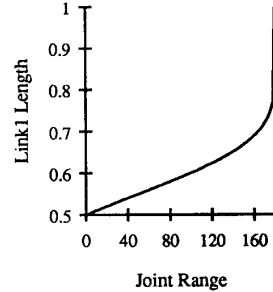


Figure 3: A plot of optimal length of link 1 versus joint range of $\phi$ for the standard shoulder and elbow joint

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