Nonholonomic Control Based on Approximate Inversion

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August 19, 1999

Abstract

A variety of simple mechanical systems are known to exploit nonholonomic effects to achieve their goals. Theories describing the operation of such systems have been developed and applied, especially in the study of robotics. Not surprisingly, the inclusion of dynamical effects adds complexity and changes some of the qualitative properties. In this paper we give a complete analysis of the optimal positioning problem for a natural class of dynamical systems whose inertial effects are linear but whose kinematics are those of the standard nonholonomic integrator. Based on these results we develop a suitable modification of the approximate inverse method for solving tracking and stabilization problems.

1 Introduction

Over the last decade it has become more widely appreciated that nonholonomic systems, rather than being something to be dismissed as a footnote in a large book about holonomic systems, are actually of considerable significance–in some sense the most intrinsically nonlinear systems in common use. The justification for studying them goes far beyond their mathematical beauty, extending to such domains as robotics [9], [10], control of molecular dynamics [7], nuclear magnetic resonance imaging, and rotating electrical machinery [11]. In a number of areas, the key ideas are brought out by the properties of the now familiar *nonholonomic integrator*

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned} \tag{1}$$

However, this model is almost always obtained as an approximation to a higher order system. The nature of the simplification is such that one can no longer formulate certain interesting questions such as those involving minimum energy gaits for locomotion, optimal pumping frequencies, etc.

[†]This work was supported in part by the US Army Research Office under grants DAAG-55-97-1-0114 (Center for Dynamics and Control of Smart Structures) and DAAL-03-92-G-0115 (Center for Intelligent Control Systems).

In this paper we apply the calculus of variations to identify the inputs which produce dynamical repositioning with minimum effort, measured in a quadratic sense. Previous work involving dynamic nonholonomic systems and control of such systems can be found in [1], [2], [6], [11], and [15]. The results here seem to be the first general optimal control results available for nonholonomic systems with a drift term. New results given here include the fact that when one cascades a linear system with nonholonomic integrators, the qualitative properties of the optimal solution depends strongly on whether the linear system precedes or follows the nonholonomic integrator. In one case the Euler-Lagrange equations can be reduced to a family of linear time invariant equations, whereas in the other case time variation cannot be avoided, and the optimal trajectories must be described using hypergeometric functions. Even so, our main interest is to extend the idea of an approximate inverse to this setting. It has previously been shown that the approximate inverse provides a conceptually clear way to approach tracking and stabilization problems associated with the nonholonomic integrator. The work presented shows that the approximate inverse retains its usefulness in the more complex situations encountered here.

2 Properties of the Approximate Inverse

In their 1991 paper [14] Sussmann and Liu showed that under some mild conditions if the system

$$\dot{x} = G(x)u \tag{2}$$

is first bracket controllable (the vector fields and their first Lie brackets span the tangent space), then one can come arbitrarily close to tracking a desired trajectory in x. Of course the notable fact is that this result holds even when the dimensionality of x exceeds that of u. For such systems, averaging methods [3], [5], [8], [10], [12], [13], and [14] have been shown to be of use in the design of controllers. In Brockett [4] it was shown that in the case of the nonholonomic integrator, one could conveniently characterize a one-parameter family of approximate inverses, and that by precompensating the system with an approximate inverse one arrives at a system which is approximately linear and time invariant over a range of frequencies $\omega \in [0, \omega_c]$ and that ω_c can be made as large as one likes. The solutions of an optimal repositioning problem can be used to guide the design of the approximate inverse. To motivate the following work in this paper, we will give a brief overview of this method for the system in Eqns. 1.

If we would like to find trajectories $x(t)$, $y(t)$ and $z(t)$ such that $x(0) = x_0$, $x(1) = x_1$, $y(0) = y_0$, $y(1) = y_1$, $z(0) = z_0$ and $z(1) = z_1$ with minimal control cost in a quadratic sense, we can define the minimization quantity

$$\eta = \int_0^1 \dot{x}^2 + \dot{y}^2 + \mu(x\dot{y} - y\dot{x})dt. \tag{3}$$

The Euler-Lagrange equations for this problem have the solutions

$$\begin{aligned} x(t) &= \alpha_1 + \alpha_3(1 - \cos(\mu t)) + \alpha_4 \sin(\mu t) \\ y(t) &= \alpha_2 - \alpha_4(1 - \cos(\mu t)) + \alpha_3 \sin(\mu t) \end{aligned} \tag{4}$$

From here we can see that if we choose the boundary conditions on x and y to be such that $x_0 = x_1$ and $y_0 = y_1$, the optimal motion of x and y for producing motion in z is with pure sinusoids. Note that although the original problem requires the matching of six boundary conditions with six parameters, only five parameters are present in the optimal solutions. The sixth parameter, however, is not necessary, because only the difference $z(1) - z(0)$ enters the problem formulation. In other words, given a starting value z_0 we can choose μ to produce the appropriate change in z to reach z_1 in the appropriate amount of time.

This knowledge of optimal control has led to the development of approximate inverses by Brockett [4] and Struemper and Krishnaprasad [12], [13]. The general form of the approximate inverse for the system discussed above is to let the controls be

$$\begin{aligned} u(t) &= \dot{\bar{x}}(t) + \sqrt{\frac{m(t)\omega}{\sin(2\phi)}}\sin(\omega t + \phi) \\ v(t) &= \dot{\bar{y}}(t) + \sqrt{\frac{m(t)\omega}{\sin(2\phi)}}\sin(\omega t - \phi) \end{aligned} \tag{5}$$

where $m(t) = (\bar{y}\dot{\bar{x}} - \bar{x}\dot{\bar{y}} + \dot{\bar{z}})$ and $\phi = sgn(m)\pi/4$. For these choices it has been shown that $x(t) \approx \bar{x}(t)$, $y(t) \approx \bar{y}(t)$ and $z(t) \approx \bar{z}(t)$.

3 Cascades and Optimal Repositioning

Relative to the two-input three-output nonholonomic integrator, one has two more or less natural classes of systems obtained by cascading with a linear time invariant system. These classes take the form (Fig. 1)

$$\begin{aligned} \dot{x} &= Ax + Bu \quad ; \qquad y = Cx \\ \dot{\xi}_1 &= y_1 \\ \dot{\xi}_2 &= y_2 \\ \dot{\xi}_3 &= \xi_1 y_2 - \xi_2 y_1 \end{aligned} \tag{6}$$

and (Fig. 2)

$$\begin{aligned} \dot{\xi}_1 &= u_1 \\ \dot{\xi}_2 &= u_2 \\ \dot{\xi}_3 &= \xi_1 u_2 - \xi_2 u_1 \\ \dot{x} &= Ax + B\xi \quad ; \qquad y = Cx \end{aligned} \tag{7}$$

We will consider the specific cases of these transformations where $A = \alpha I$, $B = I$, and $C = I$. The corresponding block diagrams are shown below with the second order system equations. In the case where the nonholonomic integrator follows the transformation (Fig. 1), we will refer to the system as a right dynamic nonholonomic integrator which will have the associated second order equations

$$\begin{aligned} \ddot{x} + \alpha\dot{x} &= u \\ \ddot{y} + \alpha\dot{y} &= v \\ \ddot{z} + \alpha(x\dot{y} - y\dot{x}) &= xv - yu \end{aligned} \tag{8}$$

In the case of the integrator preceding the transformation (Fig. 2), we will refer to the system as a left dynamic nonholonomic integrator which has the associated second order equations

$$\begin{aligned}\ddot{x}+\alpha\dot{x} &= u\\ \ddot{y}+\alpha\dot{y} &= v\\ \ddot{z}+\alpha\dot{z} &= \dot{x}v-\dot{y}u+\alpha(xv-yu)\end{aligned} \tag{9}$$

From the point of view of nonlinear control, the most important change that these transformations

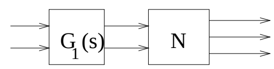

Figure 1: Right dynamic nonholonomic integrator system.

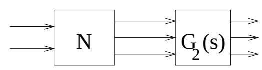

Figure 2: Left dynamic nonholonomic integrator system.

bring about is the introduction of a drift term, resulting in a set of equations of the general form

$$\dot{x}=f(x)+G(x)u \tag{10}$$

Although the drift complicates matters, the Lie algebra generated by the vector fields $\{f,g_1,...g_k\}$ is solvable, assuring that the system will remain reasonably tractable except in the case where $\alpha=0$ where we can state

Lemma: 1 : *The left cascade system is controllable whereas the right cascade system is not.*

Sketch of Proof: For the left system, the space spanned by the vectors

$$g_1,g_2,[f,g_1],[f,g_2],[g_1,g_2],[f,[g_1,g_2]] \tag{11}$$

has dimension six at all points of the state space and thus the system is completely controllable.

For the right system, the Lie algebra is formed from the vector fields

$$f,g_1,g_2,[f,g_1],[f,g_2],[[f,g_1],[f,g_2]] \tag{12}$$

where all other brackets are either identically zero or are linear combinations of these brackets. Computing the determinant of the matrix formed by these six vectors shows that in this situation we do not have a six dimensional span of the tangent space at any point of the state space. The time derivative of the quantity $\dot{z}-(x\dot{y}-y\dot{x})$ is $\ddot{z}-(xv-yu)=0$. Rewriting these equations eliminating the equation for $\ddot{z}$ results in a controllable system. The right dynamic integrator is only controllable on the submanifold defined by $\dot{z}-x\dot{y}+y\dot{x}=k$.

3.1 Optimal Trajectories for Right Dynamic Systems

Given the dynamic nonholonomic integrator in Fig. 1 with $G(s) = \frac{1}{s}I$, i.e.

$$\begin{aligned} \ddot{x} &= u \\ \ddot{y} &= v \\ \ddot{z} &= xv - yu \end{aligned} \tag{13}$$

consider minimizing the quantity

$$\eta = \int_0^1 (u^2(t) + v^2(t))dt + \mu\dot{z}(1) + \nu z(1) \tag{14}$$

Although one can introduce costates and use the maximum principal to derive the optimal trajectories, it is very much more transparent to set up the variational equations using an integration by parts

$$\int_0^1 \int_0^t (x\ddot{y} - y\ddot{x})d\tau dt = \int_0^1 (x\ddot{y} - y\ddot{x})dt - \int_0^1 t(x\ddot{y} - y\ddot{x})dt \tag{15}$$

This allows us to work with an equivalent problem defined by

$$\eta = \int_0^1 (\ddot{x}^2 + \ddot{y}^2 + (\tilde{\mu} + \tilde{\nu}t)(x\ddot{y} - y\ddot{x}))dt \tag{16}$$

The corresponding Euler-Lagrange equations are

$$\begin{aligned} x^{(4)} - \tilde{\mu}y^{(1)} &= 0 \\ y^{(4)} + \tilde{\mu}x^{(1)} &= 0 \end{aligned} \tag{17}$$

Solving these equations with $\omega = \tilde{\mu}^{\frac{1}{3}}$ leads (see Fig. 3) to the general form

$$\begin{aligned} x(t) &= \alpha_1 + \alpha_3\cos(\omega t) + \alpha_4\sin(\omega t) \\ &\quad + \frac{1}{2}\cos(\frac{1}{2}\omega t)\left(\alpha_5 e^{-\frac{\sqrt{3}}{2}\omega t} + \alpha_6 e^{\frac{\sqrt{3}}{2}\omega t}\right) \\ &\quad + \frac{1}{2}\sin(\frac{1}{2}\omega t)\left(\alpha_7 e^{-\frac{\sqrt{3}}{2}\omega t} + \alpha_8 e^{\frac{\sqrt{3}}{2}\omega t}\right) \\ y(t) &= \alpha_2 - \alpha_4\cos(\omega t) + \alpha_3\sin(\omega t) \\ &\quad + \frac{1}{2}\cos(\frac{1}{2}\omega t)\left(\alpha_7 e^{-\frac{\sqrt{3}}{2}\omega t} + \alpha_8 e^{\frac{\sqrt{3}}{2}\omega t}\right) \\ &\quad - \frac{1}{2}\sin(\frac{1}{2}\omega t)\left(\alpha_5 e^{-\frac{\sqrt{3}}{2}\omega t} + \alpha_6 e^{\frac{\sqrt{3}}{2}\omega t}\right) \end{aligned} \tag{18}$$

where the coefficients α_i can be solved as functions of the boundary conditions on x, $\dot{x}$, y, and $\dot{y}$. From the lemma, we know that the value $\dot{z}$ is determined by the initial data. We are then left with a choice for the parameter $\tilde{\mu}$ which allows us to meet the endpoint conditions on z as discussed in the previous section.

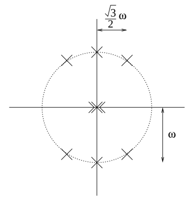

Figure 3: Eigenvalues for optimal solutions of right dynamic system.

3.2 Optimal Trajectories for Left Dynamic Systems

In this case we have $\ddot{z} = \dot{x}\ddot{y} - \dot{y}\ddot{x}$ and we would like to minimize the quantity

$$\eta = \int_0^1 \frac{1}{2}\ddot{x}^2 + \frac{1}{2}\ddot{y}^2 + (\mu + t\nu)(\dot{x}\ddot{y} - \dot{y}\ddot{x})dt \tag{19}$$

The Euler-Lagrange equations are

$$\begin{aligned} x^{(4)} - 2(\mu + t\nu)y^{(3)} - 3\nu y^{(2)} &= 0 \\ y^{(4)} + 2(\mu + t\nu)x^{(3)} + 3\nu x^{(2)} &= 0 \end{aligned} \tag{20}$$

Using the change of variables $a = x + iy$ we have

$$a^{(4)} = -2i(\mu + \nu t)a^{(3)} - 3i\nu a^{(2)} \tag{21}$$

where

$$\frac{d^2}{dt^2}(\ddot{a} + 2i(\mu + \nu t)\dot{a} - i\nu a) = a^{(4)} + 2i(\mu + \nu t)a^{(3)} + 3i\nu a^{(2)}. \tag{22}$$

With the further transformation $b = ae^{\gamma(t)}$ where $\gamma(t) = i(\mu t + \nu t^2/2)$, Eqn. 21 reduces to

$$\ddot{b} = ((\mu + \nu t)^2 + 2i\nu)b. \tag{23}$$

The general solution for this equation involves hypergeometric functions [16]. We note, in passing, that in the special case $\nu = 0$ where we ignore the constraint on the value of $z(1)$, we eliminate the imaginary components of the parameters in Eqn. 23. In this situation, we can go back to the original Euler-Lagrange equations for this system and directly solve them to find that they involve polynomial functions of time together with products of hyperbolic sinusoids and complex exponentials.

Although there does not exist a closed form solution for the full optimization problem, we can determine important aspects of the qualitative properties of the solution by numerically simulating the Euler-Lagrange equations. As an example, choosing $\mu = 1$, $\nu = 2$, and the initial conditions $[x, x^{(1)}, x^{(2)}, x^{(3)}, y, y^{(1)}, y^{(2)}, y^{(3)}] = [0, 0, 0.1, 0.1, 0, 0, 0, 0]$, the trajectory for $x(t)$ and $y(t)$ are shown in Fig. 4. For this system, the controls are simply the second derivatives of x and y, and are shown in Fig. 5.

4 Approximate Inversion

Motivated by earlier work with approximate inversion and the results from the previous section, we are led to a design for a stabilizing controller for the left dynamic nonholonomic system. Although

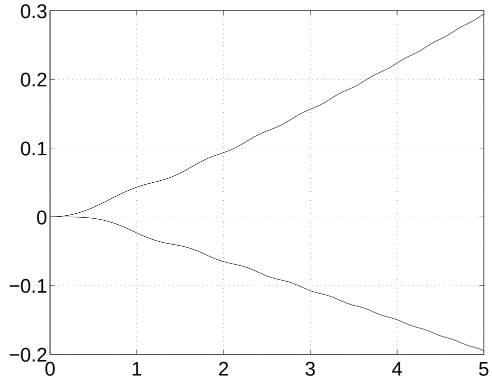

Figure 4: Simulated solution of x and y states of Euler-Lagrange equations for left dynamic system.

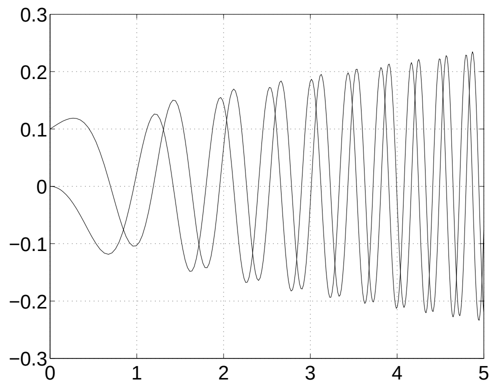

Figure 5: Optimal controls from Euler-Lagrange equations for left dynamic system.

we do not have a closed form solution for the optimal repositioning problem for this system, in the case where we are not constraining the value of $z(1)$, we know that the optimal solutions are approximately sinusoidal. Consider then, the controller defined by

$$\begin{aligned} u(t) &= \ddot{\bar{x}}(t) + m(t)\sin(\omega t + \phi) \\ v(t) &= \ddot{\bar{y}}(t) + m(t)\sin(\omega t - \phi) \end{aligned} \tag{24}$$

Integrating u and v once and using integration by parts for the sinusoidal terms, an application of the Riemann-Lebesgue theorem shows that for ω large, $\dot{x}$ and $\dot{y}$ are approximately

$$\begin{aligned} \dot{x}(t) &\approx \dot{\bar{x}}(t) - \frac{1}{\omega}m(t)\cos(\omega t + \phi) \\ \dot{y}(t) &\approx \dot{\bar{y}}(t) - \frac{1}{\omega}m(t)\cos(\omega t - \phi) \end{aligned} \tag{25}$$

Calculating $\ddot{z}$ then gives

$$\ddot{z}(t) \approx \dot{\bar{x}}\ddot{\bar{y}} - \dot{\bar{y}}\ddot{\bar{x}} + \frac{1}{\omega}m^2(t)\sin(2\phi) \tag{26}$$

This approach allows us to specify trajectories for three of the six states of the system. However, for purposes of stabilization, specifying that $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ be damped sinusoids

$$\begin{aligned} \ddot{x} &= -k_1 x - k_2 \dot{x} \\ \ddot{y} &= -k_1 y - k_2 \dot{y} \end{aligned} \tag{27}$$

with

$$\begin{aligned} d(t) &= -k_1 z - k_2 \dot{z}, \qquad \phi = sgn(d(t))\frac{\pi}{4} \\ m(t) &= \sqrt{\frac{d(t)\omega}{\sin(2\phi)}} \end{aligned} \tag{28}$$

will result in stabilization of the system. Figs. 6 and 7 show the stabilization results for $k_1 = 2$, $k_2 = 0.2$, $\omega = 10$ and $\omega = 100$. As we would expect, increasing ω decreases the error between the

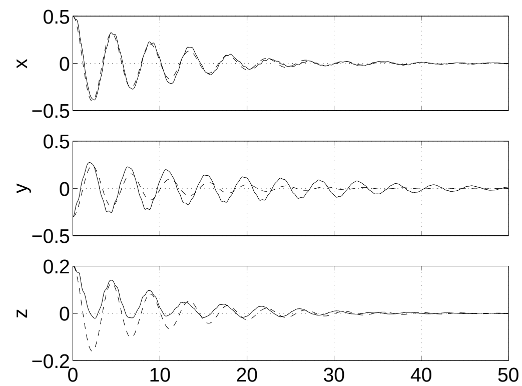

Figure 6: Left dynamic system stabilization with $\omega = 10$.

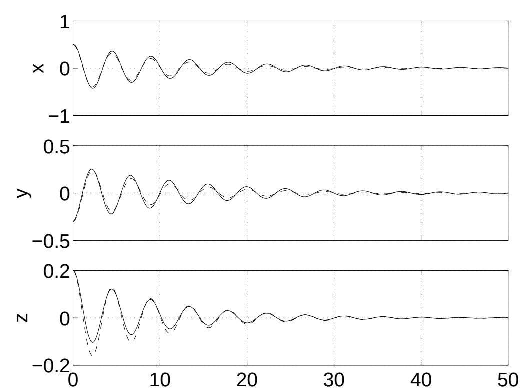

Figure 7: Left dynamic system stabilization with $\omega = 100$.

states and the trajectory of the damped sinusoid $\ddot{w} = -2w - 0.2\dot{w}$ which is represented by the dashed line in each plot.

5 Conclusion

In this paper we have generalized the well-known kinematic nonholonomic integrator to a dynamic setting through the cascade of the nonholonomic integrator with a choice of linear time invariant systems. The placement of integrators relative to the nonholonomic integrator has been shown to lead to classes of second order systems which can be distinguished by their controllability properties. Using the calculus of variations, optimal solutions for the repositioning problem were determined. These results were then applied to the problem of approximate inversion.

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