Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

Minimum Attention Control

R. W. Brockett [1]

Abstract

In this paper we consider the question of how to trade-off the complexity of implementing a control law against the performance of the control system. We propose a measure of complexity and formulate optimization problems involving both a measure of the desirability of the trajectories and the difficulty of implementing the control law. By introducing criteria of the type proposed here it is possible to strike a balance between the performance of the system and the difficulty involved in implementing the control. The optimization problems coming from this formulation are similar to those arising in the variational description of fields in physics.

1 Introduction

Although much of the literature on automatic control is concerned with definitions of optimality relating to the shape of trajectories and the magnitude of the control required to generate them, it often happens that costs related to the implementation of the control are more significant than the cost associated with the trajectories. In biological motor control, for example, there are many situations in which imprecise control is virtually as good as precise control; the more important objective is to find a control law that can be implemented without diverting attention from the other tasks which are more pressing. This raises the question, "How can the cost of implementation be taken into account when formulating an optimal control problem?"

Our point of view is that the easiest control law to implement is a constant input. Anything else requires some attention. The more frequently the control changes, the more effort it takes to implement it. Because the control law will depend on the state x and the time t, it can be argued that the cost of implementation is linked to the rate at which the control changes with changing values of x and t. This rate of change may also affect the effort required to compute the desired control or some suitable approximation to it. In any case solutions that require less frequent adjustments as x and t change are to be preferred over those that require more frequent adjustments. From the point of view of an animal controlling its body, or a systems engineer allocating the cpu cycles of a computer controlling a machine tool, control laws with small values of $||\partial u/\partial t||$ and $||\partial u/\partial x||$ require less frequent updating and will be more robust with respect to small changes in the data. These considerations suggest that a suitable quantification of what is meant by "attention" might be be obtainied by squaring the values of the partial derivatives, adding them and integrating over all time and space to get a measure of the attention required to implement a given control law.

This reasoning suggests a class of optimization problems associated with selecting the architecture of a control system. The general structure of the optimization problem will involve minimizing functionals of the form

$$\eta_a = \int_\Omega \phi\left(x, t, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) dxdt$$

subject to constraints on u such as will insure that the performance is adequate for the task. We will refer to η as an *attention functional.*

Textbooks on optimal control discuss the difference between open-loop and closed-loop control however the classification is rather informal and in many cases (e.g., fixed end-point linear-quadratic optimal control on finite time intervals) it is unclear what might be meant by a closed-loop solution. This makes it difficult for researchers in other fields to discuss the distinction in a precise way. At an intuitive level, it seems that biological motor control involves not only "pure" open-loop control but also a gradation of modalities spanning a range between open-loop and closed-loop operation. Intuitively, one thinks that large values of $||\partial u/\partial x||$ indicate closed-loop control and that large values of $||\partial u/\partial t||$ indicate open-loop control. By modifying the attention functional we can change the ratio of the penality put on the closed-loop $||\partial u/\partial x||$ terms relative to the penality put on the open-loop $||\partial u/\partial t||$ terms. In this way we create a continiuum and arrive at a characterization which makes possible a quantitative study of the trade-offs between open-loop and closed-loop control.

We remark in passing that the way we formulate the minimum attention stabilization problem here yields equations that are similar to the equations describing

[1] Army DAAH 04-93 -G- 0330, UMD-SRC NSF EEC 94-02384, Brown Army DAAH 04-96-1-0445, Draper Learning, Airforce F33615-95-C-1759, JSEP, Navy N00014-89-J-1023 , CDCSS Army DAAG 55 97 1 0114, Draper Modeling,C.S. Draper Laboratories, DLH 496022, MIT Army DAAL03-92-G-0115

0-7803-3970-8/97 $10.00 © 1997 IEEE

fields in physics. In the variational approach one usually formulates a Lagrangian density derived from the kinetic and potential energy and obtains the field equations as the corresponding Euler-Lagrange equations. (See, for example, Abraham and Marsden [1]. who treat the the gravitational field in this spirit.) In the present circumstances the minimum attention controls will be characterized as the solutions of a set of partial differential equations involving the state variables and time.

2 Attention and Stability

Consider the problem of finding a stabilizing control law for the scalar system

$$\dot{x} = u$$

that can be implemented with as little "attention" as possible. In order to make this into a precise problem we must choose an *attention functional* which gauges the attention required to implement a particular feedback control law. In this example we seek a control law $u : [0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ having the property that all trajectories go to zero as t goes to infinity and such that the "variation" of u is not too big. To this end, we choose the attention functional

$$\eta_a = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 dtdx$$

and ask that u satisfy suitable boundary conditions. Of course we must have $u(t, 0) = 0$ if the null solution is to be an equilibrium solution. Local stability of the null solution demands that the integral of $u_x(t, 0)$ should diverge to minus infinity. That is to say, if the solution $x = 0$ is to have the property that the linearization about it

$$\dot{\delta}(t) = a(t)\delta(t) = \left.\frac{\partial u}{\partial x}\right|_{(0,t)} \delta(t)$$

is asymptotically stable we need

$$\lim_{t\to\infty} \int_0^t a(\sigma)d\sigma = -\infty$$

We will refer to this integral as the *stability integral.* Somewhat more subtle is the matter of boundary conditions on $u(0, x)$. We may think of $u(0, x)$ as being the value that u must assume when the control system is "turned on" at $t = 0$ suggesting the following argument. Before the control system is engaged there is no information on x available and so we must think of u as being held at some fixed value. Its value can only be changed on the basis of a measurement of the state and the execution of some action based on this information. This argues for the boundary condition $u(0, x) = a$. In principle, a could be any real number but if we acknowledge that $u(t, 0) = 0$ we see that $a = 0$ is the only possible choice consistent with a smooth control law. Thus we have boundary conditions dictating the value of u on the positive t axis and the entire x axis as well as a condition on the integral over time of the of the partial derivative $u_x(t, 0)$. We will see below that, specifying the the partial derivative of u with respect to x on the line $x = 0$, together with the standard first order necessary conditions "over determines" u and forces a non-smoothness into the problem.

In formulating the above attention functional we have implicitly assumed that x and t have been scaled in such a way as to be equally "significant". Introducing the metric $(ds)^2 = (dt)^2 + (dx)^2$, we can express η_a in terms of the gradient

$$\eta_a = \int_{-\infty}^{\infty} \int_0^{\infty} ||\nabla u||^2 dtdx$$

To begin with, we observe that the control law

$$u(t, x) = -\frac{t}{1+t^2}\frac{x}{1+x^2}$$

satisfies the boundary conditions described above, makes the null solution of $\dot{x} = u$ asymptotically stable and is such that the partial derivatives

$$u_t(t, x) = -\frac{1-t^2}{(1+t^2)^2}\frac{x}{1+x^2}$$

$$u_x(t, x) = -\frac{1}{(1+t^2)}\frac{1-x^2}{(1+x^2)^2}$$

are square integrable on the space

$$F = \{(t, x) : 0 \le t \le \infty; -\infty \le x \le \infty\}$$

Thus this function confirms the existence of a stablizing control requiring only a finite amount of attention. On the other hand, a linear, time-varying feedback control law $u = -k(t)x$ will meet the boundary conditions if $k(0) = 0$ but requires infinite attention relative to this measure in all but the trivial (and non stabilizing case) $k = 0$.

It is well known, and easy to demonstrate, that a function which satisfies the boundary conditions and possesses partial derivatives up to order two can be a local minimum for the functional defined by the square of the gradient only if it satisfies the associated Euler-Lagrange equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right) u(t, x) = 0$$

Under this hypothesis, a suitable integration-by-parts shows that the minimizing value of the attention functional is expressible in terms of the boundary values of u as

$$\eta_a(u) = \int_R \frac{\partial u(0, x)}{\partial t} u(0, x)dx + \int_{R^+} \frac{\partial u(t, 0)}{\partial t} u(0t, 0)dt$$

assuming that u_x becomes vanishingly small as $|x|$ goes to infinity. However, for the given boundary conditions on u we see that the square of the gradient would be zero. This is inconsistent with stability and so it must happen that the conditions required for the integration-by-parts are violated. We will not undertake a systematic study of this here but we do point out that it makes sense to search for a minimum of the attention functional within a class of u's that only have first partial derivatives almost everywhere. For example we could consider a class of functions with piecewise smooth first partials. In fact, we will show that there exist solutions to the variational equations and the boundary data in this class.

With the goal of finding a suitable u satisfying the variational equation, we introduce a real analytic function $\phi(t)$, its complex extension $\phi(t+ix)$ has an imaginary part $u(t,x) = (\phi(t+ix) - \phi(t-ix))/2i$ that vanishes on the line $x = 0$ and satisfies Laplace's equation in a neighborhood of the t-axis. The expression given above for η_a shows that we cannot find a suitable u that is twice diffrentiable everywhere and so we seek to piece together in a continuous way solutions consisting of harmonic functions defined on subsets of the domain of interest. Based on the observation that the mapping $(t,x) \rightarrow (x,t)$ sends the first quadrant and the boundary conditions on u into themselves, we look for a solution of the variational equations on the wedge interior to the two 45^o lines, $x = \pm t$. We will then extend this solution to the whole half-plane by reflecting in these lines.

The particular attention functional being considered here will only be finite if the control becomes "soft" for large values of $|x|$ and t. We have already noted that linear control is excluded. If in the wedge $|x| \leq t$ we let u be given by the imaginary part of $\phi(z) = -\log(z+k)$, k a real, positive constant, thenin that wedge

$$u(t,x) = \frac{-1}{2i}\left(\log(t+k+ix) - \log(t+k-ix)\right)$$

When extended to the whole half-space $t \geq 0$ by reflecting in the 45^o line, this choice gives

$$u(t,x) = \begin{cases} -\tan^{-1}\frac{x}{t+k} & |x| \leq t \\ -\tan^{-1}\frac{t}{x+k} & |x| \geq t \end{cases}$$

Clearly this function is continuous in the half-plane $t \geq 0$ and has a continuous derivative except on the lines $t = \pm x$. Within the 45^o wedge the derivatives are given by

$$u_t = \frac{-x}{((t+k)^2+x^2}$$

$$u_x = \frac{-t-k}{((t+k)^2+x^2}$$

Because $\log t$ goes to infinity with t the stability integral diverges as required. However the integrand of the attention functional is given by

$$||\nabla u||^2 = \frac{1}{((t+k)^2+x^2}$$

which makes the attention functional diverge. Thus this choice of control does not go to zero rapidly enough for large values of x and t and we may say that the boundary condition

$$u_x(t,0 = -\tan^{-1}\frac{t}{x+k}$$

is incompatable with the desire to minimize the given attention functional.

On the other hand, we can show that the control defined within the 45^o wedge by the imaginary part of $\phi(z) = -\log(\log z + k)$ with $k \leq 1$ that is, for $|x| \leq t$

$$u(t,x) = \frac{-1}{2i}\left(\log(\log(t+k+ix)) - \log(\log(t+k-ix))\right)$$

not only make the stability integral divergent, as required, but also keeps the attention functional finite. Starting with

$$\log(t+ix) = \log\sqrt{(t+k)^2+x^2} + i\tan^{-1}\frac{x}{t+k}$$

we see that the imaginary part of

$$\phi(t+ix) = -\log(\log\sqrt{(t+k)^2+x^2} + i\tan^{-1}\frac{x}{t+k})$$

is given by

$$u(t,x) = -\tan^{-1}\frac{\tan^{-1}\frac{x}{t+k}}{\log\sqrt{(t+k)^2+x^2}}$$

supplementing this with the expression obtained by reflection we get

$$u(t,x) = \begin{cases} -\tan^{-1}\frac{\tan^{-1}\frac{x}{t+k}}{\log\sqrt{(t+k)^2+x^2}} & |x| \leq t \\ -\tan^{-1}\frac{\tan^{-1}\frac{t}{x+k}}{\log\sqrt{(x+k)^2+t^2}} & |x| \geq t \end{cases}$$

For large values of t and/or x the control is small and the leading $\tan^{-1}$ can be dropped. In this case the control is approximated by

$$u(t,x) = \begin{cases} -\frac{\tan^{-1}\frac{2x}{t+k}}{\log((t+k)^2+x)} & |x| \leq t \\ -\frac{\tan^{-1}\frac{2t}{x+k}}{\log((x+k)^2+t^2)} & |x| \geq t \end{cases}$$

Thus we see that this control falls off faster at infinity than the $u = \tan^{-1}(x/t+k)$ control investigated previously.

This additional log in the denominator is enough to make the attention functional converge. The details

are left to the reader but the key fact is that the infinite integral

$$I = \int_a^\infty \frac{1}{x \log^k x} dx$$

while divergent for $k = 1$, converges if $k = 2$. Thus there exists a finite attention control law that drives x to zero from any initial condition.

A systematic discussion of the minimum attention control of linear systems can be organized along the following lines. If the uncontrolled system has all its eigenvalues on the imaginary axis then one can proceed by analogy with the results just given. For example, the controlled harmonic oscillator

$$\ddot{x} + x = u$$

can be stabilized by

$$u(t, \dot{x}) = \frac{-1}{2i} \left(\log(\log(z + k)) - \log(\log(\bar{z} + k))\right)$$

with $z = t + i\dot{x}$. Linear systems with poles in the right half-plane can not be stabilized by a control with η_a finite. This leaves as the remaining interesting case the situation in which the uncontrolled dynamics are asymptotically stable and the control enters as

$$\dot{x} = Ax + bu$$

We investigate this in the next section.

3 Attention and Average Performance

To fix ideas, consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu$$

and suppose that it is desired to modify a trajectory based performance of the form

$$\eta_t = \int_0^\infty x^T Q x + u^T u dt$$

by adding a term that will limit the value of the attention functional

$$\eta_a = \int_{-\infty}^\infty \int_0^\infty \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 dt dx$$

Problems of this type must be posed in terms of families of trajectories rather than individual trajectories as one does in optimal control. We achieve this by assuming that the initial state of the system is random with probability density $\rho_0(x)$ and then minimize

$$\eta = \eta_a + \mathcal{E}\eta_t$$

To put this into mathematical form we introduce an evolution equation for the density of the random variable $x(t)$,

$$\frac{\partial \rho}{\partial t} = -\langle \frac{\partial}{\partial x}, (Ax + Bu)\rho \rangle \; ; \; \rho(0, x) = \rho_0(x)$$

along with an evolution equation for u

$$\frac{\partial u(t, x)}{\partial t} = v(t, x) \; ; \; u(0, x) = 0$$

The performance measure is $\eta = \eta_t + \eta_a$ with

$$\eta_t = \int_0^\infty \int_{R^n} \rho(t, x)(x^T Q x + ||u||^2) dx dt$$

and

$$\eta_a = \int_0^\infty \int_{R^n} ||\partial u / \partial x||^2 + ||v||^2 dx dt$$

As in the stability problem of the previous section, boundary conditions must be provided. Again we can argue that $u(0, x)$ should vanish along with $u(t, 0)$. Because the open-loop system is assumed to be stable there is no analog of the condition arising from stability considerations. Because the choice $u(t, x) = 0$ gives a finite value for the performance measure, and because the performance is bounded from below by zero, there must be an infinimizing sequence of controls.

Because u is not readily expressable in terms of ρ and its derivatives, the first order necessary conditions in this case are more conveniently written in terms of a state-costate pair. To this end, we introduce the costate in the form $(\psi(t, x), \chi(t, x))$ with ψ being the partner of ρ and χ being a vector of functions whose dimension equals that of the dimension of u. The hamiltonian is then

$$h(\rho, u, \psi, \chi) = \int_{R^n} -\psi \langle \frac{\partial}{\partial x}, (Ax + Bu)\rho \rangle + \langle \chi, v \rangle + f dx$$

with f being the sum of the integrands of η_t and η_a

$$f = \rho(t, x)(x^T Q x + ||u||^2) + ||v||^2 + v^T v$$

The (formal) first order necessary conditions will then be obtained in terms of the coupled partial differential equations obtained from this hamiltonian

$$\frac{\partial \psi}{\partial t} = -\langle Ax + Bu, \frac{\partial \psi}{\partial x} \rangle + x^T Q x + u^T u$$

$$\frac{\partial \chi}{\partial t} = -\langle \frac{\partial \psi}{\partial x}, B^T \rho \rangle - u\rho$$

Of course the control v is to be choosen so as to minimize h; that is $v = -\chi$.

The role of these equations is to make concrete the qualitative question involving the trade-off between the partial of u with respect to x verses the partial of u with respect to t. When $||Ax + bu||$ is large it is more economical to use the former whereas when $||Ax + Bu||$ is small it is better to use the latter.

4 Attention and Stochastic Control

In this section we illustrate the formulation of a perfect observation stochastic control problem with a penality for attention. For the sake of comparison we again consider an asymptotically stable linear system but now with white noise excitation

$$\dot{x} = Ax + Bu + G\dot{w}$$

we look for a control u depending on x and t such that the sum of

$$\eta_t = \frac{1}{T}\mathcal{E}\int_0^T x^T Q x + u^T u dt$$

plus the value of the attention functional

$$\eta_a = \int_{-\infty}^{\infty}\int_0^{\infty}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 dtdx$$

is as small as possible. We assume that the initial state of the system is random with known density $\rho_0(x)$.

Again we introduce a pair of evolution equations in the the density ρ and the control u. If $L = L(u)$ is the Fokker-Planck operator for the given stochastic differential equation, then

$$\frac{\partial \rho(t,x)}{\partial t} = L\rho(t,x)\ ;\ \rho(0,x) = \rho_0(x)$$

$$\frac{\partial u(t,x)}{\partial t} = v(t,x)\ ;\ u(0,x) = 0$$

and express the terms in the performance measure as $\eta_t + \eta_a$ with

$$\eta_t = \int_0^{\infty}\int_{R^n} \rho(t,x)(x^T Q x + ||u||^2)dxdt$$

and

$$\eta_a = \int_0^{\infty}\int_{R^n} ||\partial u/\partial x||^2 + ||v||^2 dxdt$$

As in the stability problem, boundary conditions must be provided. Again we can argue that $u(0,x)$ should vanish along with $u(t,0)$. Because the open-loop system is assumed to be stable there is no analog of the condition arising from stability considerations.

5 Descretization

The above models based, as they are, on analysis and precise real numbers do not speak directly to the issue of computer implementation. Even so, we can use them to guide the selection of feedback controls that are well adapted to computer implementation. Introduce the notation

$$\lfloor x \rfloor = \text{greatest integer in x}$$

and observe that for α a positive number and u given by $u = -\tan^{-1}(x/t+1)$ the descritized function

$$u_d(t,x) = \begin{cases} \frac{1}{\alpha}\lfloor \alpha u(t,x)\rfloor & u(t,x) \geq 0 \\ \frac{-1}{\alpha}\lfloor -\alpha u(t,x)\rfloor & u(t,x) \leq 0 \end{cases}$$

has, for α large, the same general shape as u while only taking on only a finite number of values. It does not produce an asymptotically stable null solution of course, no control assuming a finite number of values can, but as α becomes larger it does approximate arbitrarily well in the uniform topology a smooth control which produces asymptotic stability. This is possible because of the saturating nature of the smooth control law that emerged in the stability problem. No comparable statement can be made about a linear control law.

6 References

1. Ralph Abraham and Jerrold Marsden, *Foundations of Mechanics,*. Benjamin/Cummings, Reading, Ma. 1978

2. Wing Wong and R. W. Brockett, "Systems with finite bandwidth constraints I: State Estimation Problems," *IEEE Trans. Automatic Control*, to appear.

3. Wing Wong and R. W. Brockett, Systems with Finite Communication Bandwidth Constraints II: Feedback Control Problems, *IEEE Trans. Automatic Control*, to appear.

4. R. W. Brockett, "Stabilization of Motor Networks," *Proceedings of the 34th IEEE Conference on Decision and Control*,(1995).

5. R. W. Brockett, "Characteristic Phenomena and Model Problems in Nonlinear Control," *Proceedings of the 1996 IFAC Congress* Vol. G, pp. 135-140.(1996).