ASYMPTOTIC STABILITY AND FEEDBACK STABILIZATION

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Abstract. We consider the local behavior of control problems described by ($\dot{x} = dx/dt$)

$$\dot{x} = f(x,u) \quad ; \quad f(x_o,0) = 0$$

and more specifically, the question of determining when there exists a smooth function $u(x)$ such that $x = x_o$ is an equilibrium point which is asymptotically stable. Our main results are formulated in Theorems 1 and 2 below. Whereas it might have been suspected that controllability would insure the existence of a stabilizing control law, Theorem 1 uses a degree-theoretic argument to show this is far from being the case. The positive result of Theorem 2 can be thought of as providing an application of high gain feedback in a nonlinear setting.

1. Introduction

In this paper we establish general theorems which are strong enough to imply, among other things, that

a) there is a continuous control law $(u,v) = (u(x,y,z),v(x,y,z))$ which makes the origin asymptotically stable for

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = xy$$

and that

b) there exists no continuous control law $(u,v) = (u(x,y,z), v(x,y,z))$ which makes the origin asymptotically stable for

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = yu - xv$$

The first of these implies that the null solution of Euler's angular velocity equations can be made asymptotically stable with two control torques aligned with principle axes. (See [1,2] for a general discussion, but not this particular result.) The second provides a counter example to the oft repeated conjecture asserting that a reasonable form of local controllability implies the existence of a stabilizing control law. Section 2 gives certain background material in control theory. In section 3 we formulate our nonexistence result and in section 4 we give a criterion for the existence of stabilizing control laws.

Sussmann [3] gives an example of a system in $\mathbb{R}^2$ which is controllable in a strong sense and yet fails to have a continuous feedback control law yielding global asymptotic stability. His example involves both bounds on the controls and nonlocal considerations, ours involves neither.

2. Control Systems

We intend to work locally in this paper, but even so it is perhaps worthwhile to make a few remarks about a global formulation. A more detailed and systematic account can be found in [3], but in any case the reader familiar with control theory can go directly to section 3.

Let X be a differentiable manifold and let $\pi : E \to X$ be a vector bundle over X. Let TX denote the tangent bundle of X and let π^*TX denote the pullback of TX over E. A section of π^*TX is then an assignment of a velocity vector in TX for each point in E. If we choose a local trivialization of E and pick coordinates then a section $\gamma \in \Gamma(E, \pi^*TX)$ is equivalent (in an obvious notation) to

$$\dot{x} = f(x,u)$$

such a γ is called a <u>control system</u>.

A section $\alpha \in \Gamma(X,E)$ is an assignment of a pair (u,α) corresponding to each x and so locally it is given by a function $\alpha(x)$. We denote by γ^α the section of $\Gamma(E,\pi^*TX)$ defined in coordinates by

$$\dot{x} = f(x, u + \alpha(x))$$

and say that γ^{α} is obtained from γ by the application of the feedback control law α. Associated with every control system γ there is a vector field which is obtained by setting $u = 0$. This vector field will be denoted by γ_o and is called the drift. We now state a precise problem. Given $\gamma \in \Gamma(E, \pi^*TX)$ and $x_o \in X$ under what circumstances does there exist $\alpha \in \Gamma(X,E)$ such that x_o is an equilibrium point of γ_o^{α} which is locally asymptotically stable. We call this the local feedback stabilization problem.

Now the fibers of E and the fibers of π^*TX are both vector spaces and so it makes sense to ask if the mapping of $\Gamma(X,E) \times \Gamma(E,\pi^*TX) \to \Gamma(E,\pi^*TX)$ defined by $(\alpha,\gamma) \longmapsto \gamma^{\alpha}$ is affine with respect to α. If so we call the system input-linear.

Observe that input-linear systems have a local description of the form

$$\dot{x}(t) = f(x) + \Sigma u_i g_i(x)$$

By a standard linear control system we understand a section $\gamma \in \Gamma(E,\pi^*TX)$ with $X = \mathbb{R}^n$, $E = \mathbb{R}^m \times \mathbb{R}^n$ (the trivial bundle) with γ being given by

$$\dot{x} = Ax + \sum_{I+1}^{m} b_i u_i$$

3. Nonexistence of Stabilizing Control Laws

There is one situation in which the stabilization problem has been completely understood for some time. We formulate the results here in such a way as to provide a guide to the latter developments.

Remark: Consider the standard linear control system

$$\dot{x} = Ax + Bu \quad ; \quad x(t) \in \mathbb{R}^n \qquad (*)$$

A necessary and sufficient condition for there to exist a control law α which makes $x = 0$ an asymptotically stable equilibrium point is that there exists a neighborhood N of $x = 0$ such that for each $\xi \in N$ there is a function of time $u_{\xi}(\cdot)$ defined on $[0,\infty)$ such that the solution of (*)

with initial condition $x(0) = \xi$ and control $u(\cdot) = u_\xi(\cdot)$ goes to zero as t goes to infinity.

Proof: This condition is clearly a necessary one since solutions starting near an asymptotically stable equilibrium point must approach it as t goes to infinity. To see that this condition is sufficient one observes that Range $(B, AB, \ldots, A^{n-1}B)$ is an invariant subspace for A and by change of basis (*) can be written as

$$\begin{bmatrix} \dot{x}_u \\ \dot{x}_\ell \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_u \\ x_\ell \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u$$

with Range $(B_u, A_{11}B_u, \ldots, A_{11}^{n-1}B_u) = \dim x_u$. Clearly the eigenvalues of A_{22} must have negative real parts if x_ℓ is to go to zero as t goes to infinity. On the other hand, it is well known and easily proven that if $(B, AB, \ldots, A^{n-1}B)$ is of rank n then there exists an m by n matrix K such that $(A + BK)$ has its eigenvalues in the open left half-plane. The remark then follows.

The rank condition just mentioned is necessary and sufficient for $\dot{x} = Ax + Bu$ to have a certain controllability property. If for any given pair x_1 and x_2 and any given $T > 0$, there is a control $u(\cdot)$ defined on $[0,T]$ such that $u(\cdot)$ steers (*) from x_1 at $t = 0$ to x_2 at $t = T$ we say that the control system (*) is controllable. The rank condition is necessary and sufficient for controllability in this sense. A short summary of all this reads as follows. <u>The null solution of (*) is stabilizable if and only if all the modes associated with eigenvalues with non-negative real parts are controllable</u>.

It is, in view of this background, not completely unreasonable to hope that for nonlinear systems such as

$$\dot{x} = f(x) + \Sigma u_i g_i(x)$$

something similar might happen. A specific question in this direction is, "If every initial state in a neighborhood of x_o can be steered to x_o by a control defined on $[0,\infty)$ does there necessarily exist a feedback control law which makes x_o asymptotically stable?" In this paper we show that the answer is no provided that we want a control law with some smoothness. In general we need something more than just a controllability condition since, as we will see, the controls which steer the trajectors to zero cannot always be patched together in a smooth way.

We mention one more well known fact. If we have

$$\dot{x} = f(x,u) \quad ; \quad f(0,0) = 0$$

with $f(\cdot,\cdot)$ continuously eifferentiable with respect to both arguments, and if we define $A = (\partial f/\partial x)_o$ and $B = \partial f/\partial u$, then the control system

$$\dot{x} = Ax + Bu$$

is called the linearized system at (0,0). If the linearized system satisfies the condition Rank $(B, AB, \ldots, A^{n-1}B) = n$ then there exists a linear control law $u = Kx$ such that $A + BK$ has its eigenvalues in the open left half-plane. Moreover, if we recall that $\dot{x} = \tilde{f}(x)$ with $\tilde{f}(0) = 0$ has 0 as an asymptotically stable equilibrium point provided the eigenvalues of $(\partial f/\partial x)$ have real parts which are negative, then we see that feedback stabilization is possible for $\dot{x} = f(x,u)$ provided the linearized system is controllable. In view of the previous discussion this can be stated still more precisely. <u>There exists a stabilizing control law for</u> $\dot{x} = f(x,u)$ <u>with</u> $f(0,0) = 0$ <u>provided the unstable modes of the linearized system are controllable and there exists no stabilizing control law if the linearized system has an unstable mode which is uncontrollable</u>. The negative result here depends on the well known result of Liapunov asserting that an equilibrium point is unstable if $(\partial f/\partial x)$ at that point has any eigenvalue with a real part which is positive.

From these remarks we see that insofar as local asymptotic stability is concerned, the only difficult problems involve cases where $(\partial f/\partial x)$ has eigenvalues on the imaginary axis which correspond to uncontrollable modes of the associated linearized systems and all other uncontrollable modes of the linearized system correspond to asymptotically stable behavior. In the study of stability, the cases where $(\partial f/\partial x)$ has one or more eigenvalues with a real part which vanishes at the equilibrium point are called <u>critical cases</u>. The study of the critical cases is still far from complete. (See, e.g. the remarks of Arnold in [5], page 59). We are then, in this paper, primarily concerned with understanding certain features of the critical.

cases.

The following theorem gives a necessary condition for the existence of a stabilizing control law which, under (i) and (ii) summarizes our previous discussion and in (iii) introduces a new element which is decisive as far as large class of problems, including the second example of the introduction, are concerned.

<u>Theorem 1</u>: Let $\dot{x} = f(x,u)$ be given with $f(x_o,0) = 0$ and $f(\cdot,\cdot)$ continuously differentiable in a neighborhood of $(x_o,0)$. A necessary condition for the existence of a continuously differentiable control law which makes $(x_o,0)$ asymptotically stable is that:

(i) the linearized system should have no uncontrollable modes associated with eigenvalues whose real part is positive.

(ii) there exists a neighborhood N of $(x_o,0)$ such that for each $\xi \in N$ there exists a control $u_\xi(\cdot)$ defined on $[0,\infty)$ such that this control steers the solution of $\dot{x} = f(x,u_\xi)$ from $x = \xi$ at $t = 0$ to $x = x_o$ at $t = \infty$.

(iii) the mapping

$$\gamma : A \times \mathbb{R}^m \to \mathbb{R}^n$$

defined by $\gamma : (x,u) \longmapsto f(x,u)$ should be onto an open set containing 0.

<u>Proof</u>: We prove the necessity of (iii), the necessity of (i) and (ii) having been explained above. If x_o is an equilibrium point of $\dot{x} = a(x)$ which is asymptotically stable we know from the work of Wilson [6] that there exists a Liapunov function v such that v is positive for $x \neq x_o$, vanishes at x_o, is continuously differentiable, and has level sets $v^{-1}(\alpha)$ which are homotopy spheres. The compactness of $v^{-1}(\alpha)$ implies that there exists α and $\epsilon > 0$ such that on $v^{-1}(\alpha)$, $||\partial v/\partial x|| < 1/\epsilon$ and $\langle \partial v/\partial x, a(x)\rangle < -\epsilon$. This implies that if $||\xi||$ is sufficiently small the vector field associated with $\dot{x} = a(x) + \xi$ points inward on $v^{-1}(\alpha)$. By evaluating at time $t = 1$ the solution of $\dot{x} = a(x) + \xi$ which passes through x at $t = 0$, we get a continuous map of $\{x | v(x) \leqslant \alpha\}$ into itself. Applying the Lefschetz fixed point formula we see that this map has a fixed point which must be an equilibrium point of $\dot{x} = a(x) + \xi$. Alternatively, we could use a version of the Poincaré-Hopf Theorem which applies to manifolds with boundary (see [7], page 41) to finish off the proof. This, in turn, implies that we can solve $a(x) = \xi$

for all ξ sufficiently small. Now if

$$a(x) = f(x,u(x))$$

is to have x_o as an equilibrium point, and if x_o is to be asymptotically stable, it is clearly necessary that $\xi = f(x,u)$ be solvable for ξ small.

Remark: If the control system is of the form

$$\dot{x} = f(x) + \Sigma u_i g_i(x) \quad ; \quad x(t) \in N \subset \mathbb{R}^n$$

then condition (iii) implies that the stabilization problem cannot have a solution if there is a smooth distribution D which contains $f(\cdot)$ and $g_1(\cdot),\ldots,g_m(\cdot)$ with $\dim D < n$. One further special case: If we have

$$\dot{x} = \Sigma u_i g_i(x) \quad ; \quad x(t) \in N \subset \mathbb{R}^n$$

with the vectors $g_i(x)$ being linearly independent at x_o then there exists a solution to the stabilization problem if and only if $m = n$. In this case we must have as many control parameters as we have dimensions of X. Of course the matter is completely different in the set $\{g_i(x_o)\}$ drops dimension precisely at x_o. In this sense, distributions with singularities are the only interesting kind.

Remark: There is no stabilizing control law for $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = xv - yu$. This system satisfies conditions (i) and (ii) of the theorem, but it fails to satisfy condition (iii).

4. Existence of Stabilizing Control Law

As mentioned above, in the study of asymptotic stability of the null solution of $\dot{x} = f(x)$; $f(0) = 0$ one singles out for special attention the critical cases, i.e. those for which $(\partial f/\partial x)$ has an eigenvalue with a zero real part. Liapunov showed that in a noncritical situation there always exists a quadratic function whose derivative is negative definite in a neighborhood of zero. In fact, it may be chosen so that the derivative has a leading term which is a negative definite quadratic form. In such cases, be they stable or unstable, there exists in a neighborhood of 0 a function $v(x)$ such that $\langle \partial v/\partial x, f(x) \rangle$

is negative for $x \neq 0$ and which has the further property that it remains negative definite for some reasonable class of perturbations.

Consider the control system $\dot{x} = f(x,u)$ with $f(x_o,0) = 0$. We will say that this system has <u>finite gain at</u> x_o if there exists a function $v(\cdot)$ mapping a neighborhood of x_o into $\mathbb{R}$ such that $v(x_o)$ is 0, $v(x) > 0$ for $x \neq x_o$ and for some k

$$\langle \partial v/\partial x, f(x,u)\rangle \leq -\phi(x) + k\langle u,u\rangle$$

with $\phi(x) \geq 0$ and 0 only when $x = x_o$.

<u>Remark</u>: Note that if ϕ is continuously differentiable with $f(0,0)$, $f_x(0,0)$ and $f_u(0,0)$ all zero then

$$\dot{x} = Ax + Bu + f(x,u)$$

has finite gain at zero provided the eigenvalues of A have negative real parts. (We can take $v(x) = x'Qx$ with $QA + A'Q = -I$.) More generally, if $\dot{x} = f(x)$ has zero as an asymptotically stable equilibrium point then $\dot{x} = f(x) + ug(x)$ has finite gain at zero provided that there exists a Liapunov function $v(x)$ for $\dot{x} = f(x)$ whose rate of decay satisfies $\dot{v} \leq -Mv^{p+2}$, $M > 0$, with $\langle \nabla v, g\rangle / v^{p/2}$ bounded in a neighborhood of (0,0). To establish this last assertion we note that setting $2\beta = \langle \nabla v, g\rangle / v$ allows us to write

$$\langle \nabla v, f(x)+ug(x)\rangle - ku^2 \leq [v,u] \begin{bmatrix} -Mv^p & \beta \\ \beta & -k \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}$$

and this quadratic form can be made negative definite by taking k large enough.

We now use this definition to study stability in some critical cases.

<u>Lemma</u>: Consider the coupled differential equations

$$\dot{x} = Ax + By + g(x,y)$$

$$\dot{y} = h(x,y)$$

with $g(\cdot,\cdot)$, $h(\cdot,\cdot)$, $g_x(\cdot,\cdot)$ and $g_y(\cdot,\cdot)$ all continuous in a neighborhood of (0,0). Suppose that the eigenvalues of A have negative real parts and suppose that g, h, g_x and g_y all vanish at (0,0). Let ψ be the continuous solution of $A\psi(y)+By+g(\psi(y),y)=0$ which vanishes at $y=0$. Then this pair of equations has (0,0) as an asymptotically stable critical point provided $\dot{y}=h(u-\psi(y),y)$ has finite gain at $y=0$.

Proof: Change variables according to $\tilde{x}=x-\psi(y)$, $\tilde{y}=y$. Then

$$\begin{aligned}\frac{d}{dt}\tilde{x} &= A\tilde{x}+A\psi(\tilde{y})+B\tilde{y}+g(\tilde{x}+\psi(\tilde{y}),\tilde{y})\\ &= A\tilde{x}+g(\tilde{x}+\psi(\tilde{y}),\tilde{y})-g(\tilde{x},\tilde{y})\\ &= A\tilde{x}+\tilde{g}(\tilde{x},\tilde{y})\end{aligned}$$

where $\tilde{g}$ not only vanishes together with its first derivatives at $\tilde{x}=0$. $\tilde{y}=0$ but, in fact, vanishes when $\tilde{x}=0$. We have, then

$$\frac{d}{dt}\tilde{x} = \widetilde{A}\tilde{x}+\tilde{g}(\tilde{x},\tilde{y})$$

$$\frac{d}{dt}\tilde{y} = h(\tilde{x}-\psi(\tilde{y}),\tilde{y})$$

Let η be the Liapunov function which establishes the finite gain property for the $\tilde{y}$ equation. Using the Liapunov function

$$v(x) = \alpha\tilde{x}'Q\tilde{x}+\eta(\tilde{y})$$

where $Q\widetilde{A}+\widetilde{A}'Q=-I$, we compute

$$\begin{aligned}\dot{v} = &-\alpha\langle\tilde{x},\tilde{x}\rangle+\langle Q\tilde{x},\tilde{g}(\tilde{x},\tilde{y})\rangle\\ &+\langle\nabla\eta,h(\tilde{x}-\psi(\tilde{y}),\tilde{y})\end{aligned}$$

Since $g(\tilde{x},\tilde{y})$ is second order and vanishes when $\tilde{x}$ does $\alpha\langle\tilde{x},\tilde{x}\rangle$ dominates the second term and by virtue of the finite gain hypothesis, the left-hand side is negative definite for α sufficiently large. Asymptotic stability then follows from standard Liapunov arguments.

Notice that the effect of this lemma is to reduce the study of the stability problem to the study of a lower dimensional problem (the $\tilde{y}$ equation) by elimination of the "uninteresting" noncritical part.

This can also be interpreted in terms of time scales. The solution of the $\tilde{x}$ part goes quickly to zero whereas the $\tilde{y}$ represents motion which occurs much more slowly in the manifold defined by $\tilde{x} = 0$.

Remark: If there exists a control law which makes $x = 0$ an asymptotically stable equilibrium point for

$$\dot{x} = Ax + Bu$$

then there exists a control law which makes x_o an asymptotically stable critical point, provided $Ax_o + Bu_o = 0$ can be solved for u_o. In fact, if $u = Kx$ makes $x = 0$ an asymptotically stable equilibrium point then $u = Kx + u_o$ makes x_o an asymptotically stable equilibrium point. Thus if $x = 0$ can be made asymptotically stable then there is a whole subspace $U = \{x \mid Ax \in \text{Range } B\}$ of points which can be made asymptotically stable. Incidently, in view of part (iii) of theorem 1 we see that we can apply Sard's theorem to conclude something similar for $\dot{x} = f(x,u)$. Namely, if there exists a feedback control law which makes $x = 0$ asymptotically stable then for some neighborhood N_1 of $\dot{x} = 0$ and some neighborhood N of $x = 0$ for all $\xi \in N_1$, except a possible set of measure zero, $\{(x,u) \mid f(x,u) = \xi\} \cap N$ defines a manifold in (x,u) space.

The idea behind the following theorem is that it makes sense to divide up the question of finding a stabilizing control law into two parts, one being the choice of a slow mode behavior which is asymptotically stable on a submanifold and the other being the choice of a linear control law to drive the system to the slow mode regime.

Theorem 2: Let f and g be continuously differentiable in a neighborhood of (0,0,0), and suppose they vanish at (0,0,0) together with their first derivatives. A sufficient condition for

$$\dot{x} = Ax + Fy + Bu + f(x,y,u)$$

$$\dot{y} = Gy + g(x,y,u)$$

to be stabilizable at (0,0,0) is that there exist a pair $(K, u_o(\cdot))$ such that $A + BK$ has its eigenvalues in the open left-half-plane and for ψ the continuous solution of $(A+BK)\psi(y) + Fy + Bu_o(y) + \phi(\psi(y), y, u_o(y) + K\psi(y)) = 0$ which vanishes at 0

$$\dot{y} = Gy + g(x-\psi(y), y, u_o(y)+K\psi(y))$$

has a finite gain at (0,0) with x regarded as the input.

Proof: This is an immediate application of the lemma with u being taken to be $Kx + u_o(y)$.

Remark: To apply this to the first example of the introduction, $\dot{x} = u$; $\dot{y} = v$; $\dot{z} - xy$, we let $u = -x + z$, $v = -y - z^2$. The slow mode equation is then $\dot{z} = -z^3 + x2^2 - y2 - xy$ and $\eta(z) = z^2$ shows that this equation has finite gain at 0.

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